

HOW TO COMPUTE THE WEDDERBURN DECOMPOSITION OF A FINITE-DIMENSIONAL ASSOCIATIVE ALGEBRA

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ABSTRACT. This is a survey paper on algorithms that have been developed during the last 25 years for the explicit computation of the structure of an associative algebra of finite dimension over either a finite field or an algebraic number field. This constructive approach was initiated in 1985 by Friedl and Rónyai and has since been developed by Cohen, de Graaf, Eberly, Giesbrecht, Ivanyos, Küronya and Wales. I illustrate these algorithms with the case $n = 2$ of the rational semigroup algebra of the partial transformation semigroup PT_n on n elements; this generalizes the full transformation semigroup and the symmetric inverse semigroup, and these generalize the symmetric group S_n .

INTRODUCTION

Part 1 of this survey begins by recalling the classical structure theory of finite-dimensional associative algebras over a field; the most important results are Dickson's Theorem characterizing the radical in characteristic 0, the Wedderburn-Artin Theorem on the structure of semisimple algebras, and the Wedderburn-Malcev Theorem on lifting the semisimple quotient to a subalgebra. It continues by quoting observations from Friedl and Rónyai [13] to motivate a constructive computational approach to the theory. This explicit approach requires a presentation of the algebra by a basis and structure constants, and algorithms for calculating the following: a basis for the radical of the algebra; structure constants for the semisimple quotient; a basis for the center of the semisimple quotient; a new basis for the center consisting of orthogonal idempotents; the identity matrices in the simple ideals of the quotient; an isomorphism of each simple ideal with a full matrix algebra; explicit matrices for the irreducible representations; and a subalgebra isomorphic to the semisimple quotient. This survey emphasizes characteristic 0: in this case, all calculations can be reduced to computing the row canonical form of a matrix.

Part 2 begins by introducing some classical semigroups of Boolean matrices which are natural generalizations of the symmetric group. The main example is the semigroup of partial transformations on n elements. It continues by presenting explicit calculations for $n = 2$ to illustrate the theory and algorithms of Part 1.

1. THEORY AND ALGORITHMS

1.1. Structure theory of associative algebras. We consider only associative algebras A of finite dimension over a field F . We usually assume that F is a finite

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extension of either the field \mathbb{Q} of rational numbers or the field \mathbb{F}_p with p elements (p prime); that is, an algebraic number field or a finite field. To keep the exposition as simple as possible, we often assume that $F = \mathbb{Q}$. For the classical structure theory of finite-dimensional associative algebras, our main reference is Drozd and Kirichenko [7]. For an account of the historical development, see Parshall [17].

Definition 1. [7, §2.2] A left A -module M is **semisimple** if it is isomorphic to a direct sum of simple modules. An algebra A is **semisimple** if its left regular module is semisimple. A left ideal I of A is **nilpotent** if $I^m = \{0\}$ for some $m \geq 1$. An element $x \in A$ is **strongly nilpotent** if the principal left ideal Ax is nilpotent.

Theorem 2. [7, Corollaries 2.2.5, 2.2.6] *The following conditions are equivalent: (i) A is semisimple; (ii) A contains no nonzero nilpotent left ideals; (iii) A contains no nonzero strongly nilpotent elements.*

Theorem 3. [7, Theorem 2.4.3, Corollary 2.4.5] (Wedderburn-Artin Theorem) *Every semisimple algebra Q has a unique decomposition $Q = Q_1 \oplus \cdots \oplus Q_c$ into the direct sum of simple ideals where $Q_i Q_j = \{0\}$ for $i \neq j$. Every simple algebra is isomorphic to a full matrix algebra $M_n(D)$ for some division algebra D over F .*

Definition 4. [7, §3.1] The **radical** $R(M)$ of a left A -module M consists of all $y \in M$ such that $f(y) = 0$ for every homomorphism f from M to a simple left A -module. The **radical** $R(A)$ of the algebra is the radical of the left regular module.

Theorem 5. [7, Theorems 3.1.6, 3.1.10] *The radical $R(A)$ is the set of all strongly nilpotent elements; it is a two-sided ideal and $Q = A/R(A)$ is semisimple.*

Definition 6. [7, §6.1] An algebra A over a field F is **separable** if the scalar extension $A \otimes_F K$ is semisimple for every field extension K of F .

Theorem 7. [7, Corollary 6.1.4] *Every separable algebra is semisimple; the converse holds if F is a perfect field (in particular, if $\text{char } F = 0$ or F is finite).*

Definition 8. [7, §6.2] Let $\pi: A \rightarrow Q = A/R(A)$ be the canonical surjection. A **lifting** of Q to A is a homomorphism $\epsilon: Q \rightarrow A$ such that $\pi\epsilon$ is the identity on Q . It is clear that ϵ is injective, that $\epsilon(Q)$ is a subalgebra of A isomorphic to Q , and that $A = \epsilon(Q) \oplus R(A)$ as vector spaces. Two liftings ϵ and η are **conjugate** if there is an invertible element $a \in A$ such that $\eta(x) = a^{-1}\epsilon(x)a$ for all $x \in Q$, and **unipotently conjugate** if $a = 1 + \zeta$ for some $\zeta \in R(A)$.

Theorem 9. [7, Theorem 6.2.1] (Wedderburn-Malcev Theorem) *If $Q = A/R(A)$ is separable then a lifting exists and any two liftings are unipotently conjugate.*

1.2. A constructive approach to the classical theory. As motivation for a computational approach, we quote the following passages (with slight changes) from Friedl and Rónyai [13, §§1.1, 1.2, 1.4]: “The textbook proofs of these results are not constructive. They mostly start by picking ‘any minimal [left] ideal’. But the minimal [left] ideals may not cover more than a tiny fragment of the algebra and might be quite difficult to find. ... Finding the radical and the simple factors of the [semisimple] quotient are as essential to computational algebra as factoring integers and finding composition factors are to computational number theory and group theory. ... Such results are likely to have applications to computational group theory as well since group representations are a major source of problems on matrix algebras. ... The case of commutative associative algebras generalizes

the problem of factoring polynomials over [a field] F . Indeed, let $f \in F[x]$ and let $f = g_1^{e_1} \cdots g_k^{e_k}$ where the g_i are irreducible over F . Consider the commutative associative algebra $A = F[x]/\langle f \rangle$. The radical of A comes from the ‘degeneracy’ of f , i.e. the presence of multiple factors: $R(A)$ is generated (as an ideal of A) by $h = g_1 \cdots g_k$. The quotient $A/R(A)$ is isomorphic to $F[x]/\langle h \rangle$. This in turn is the direct sum of its simple components, the fields $F[x]/\langle g_i \rangle$ ($i = 1, \dots, k$). Finding these components is equivalent to factoring f .”

1.3. Limitations of this survey. The goal of this brief survey is to present the essential ideas in enough detail that the algorithms can be translated more-or-less directly into computer programs. Therefore, some important issues are ignored, but references will be given: (i) Computational complexity: Most of the algorithms terminate in a number of steps which is a polynomial function of the size of the input. (ii) Computing the radical in characteristic p : This is much more difficult than in characteristic 0. (iii) The possibility that the minimal polynomials of central elements do not split over the base field: This seems like a severe restriction, but it is satisfied by many important examples, such as the group algebra of the symmetric group. (iv) The general case of finding a minimal left ideal in a simple ideal of the semisimple quotient: This is equivalent to computing an explicit isomorphism of the simple ideal with a full matrix algebra.

1.4. Structure constants. Since the algebra A is finite dimensional over the field F , it is completely determined by a basis $\{a_1, \dots, a_n\}$ over F and structure constants $c_{ij}^k \in F$ such that

$$a_i a_j = \sum_{k=1}^n c_{ij}^k a_k \quad (1 \leq i, j, k \leq n).$$

1.5. The radical: Dickson’s theorem. The definition of the radical does not depend on the base field, and so we can regard A as an algebra over \mathbb{Q} or \mathbb{F}_p . In characteristic 0, Dickson’s Theorem reduces finding a basis for the radical to solving a linear system. In characteristic p , the problem is more difficult; see Friedl and Rónyai [13], Rónyai [19], Cohen et al. [3]. In this survey we consider only $F = \mathbb{Q}$.

Definition 10. For $x \in A$ the **left multiplication operator** $L_x \in \text{End}_F(A)$ is $L_x(y) = xy$, and $[L_x]$ is its matrix with respect to the given basis of A .

We assume that A is unital, adjoining an identity if necessary; then the representation $x \mapsto [L_x]$ of A is faithful and A is isomorphic to a subalgebra of $M_n(F)$.

Theorem 11. [5, §65] (Dickson’s Theorem) *If $\text{char } F = 0$ and A is a subalgebra of $M_n(F)$ then x is in the radical of A if and only if $\text{trace}(xy) = 0$ for every $y \in A$.*

We use this to express the radical as the nullspace of a matrix. Let x be a linear combination of $\{a_1, \dots, a_n\}$ such that $\text{trace}(xy) = 0$ for every y . By linearity, it suffices to assume $\text{trace}(xa_i) = 0$ for $i = 1, \dots, n$. For $x_j \in F$ we have

$$\begin{aligned} x &= \sum_{j=1}^n x_j a_j \in A, & xa_i &= \sum_{j=1}^n x_j a_j a_i = \sum_{j=1}^n x_j \sum_{k=1}^n c_{ji}^k a_k = \sum_{k=1}^n \sum_{j=1}^n c_{ji}^k x_j a_k, \\ xa_i a_\ell &= \sum_{k=1}^n \sum_{j=1}^n c_{ji}^k x_j a_k a_\ell = \sum_{k=1}^n \sum_{j=1}^n c_{ji}^k x_j \sum_{m=1}^n c_{k\ell}^m a_m = \sum_{m=1}^n \sum_{j=1}^n \sum_{k=1}^n c_{ji}^k c_{k\ell}^m x_j a_m. \end{aligned}$$

Hence the matrix representing left multiplication by xa_i and its trace are as follows:

$$[L_{xa_i}]_{m\ell} = \sum_{j=1}^n \sum_{k=1}^n c_{ji}^k c_{k\ell}^m x_j, \quad \text{trace}([L_{xa_i}]) = \sum_{j=1}^n \left(\sum_{k=1}^n \sum_{\ell=1}^n c_{ji}^k c_{k\ell}^j \right) x_j.$$

Corollary 12. *The radical of A is the nullspace of the $n \times n$ matrix Δ such that*

$$\Delta_{ij} = \sum_{k=1}^n \sum_{\ell=1}^n c_{ji}^k c_{k\ell}^\ell.$$

If A is a semigroup algebra then $a_i a_j = a_{\mu(i,j)}$ and $c_{ij}^k = \delta_{\mu(i,j),k}$, and hence

$$\Delta_{ij} = \sum_{k=1}^n \sum_{\ell=1}^n \delta_{\mu(j,i),k} \delta_{\mu(k,\ell),\ell} = \sum_{\ell=1}^n \delta_{\mu(\mu(j,i),\ell),\ell}.$$

Corollary 13. (Drazin [6]) *For a semigroup algebra A we have*

$$\Delta_{ij} = |\{ \ell \mid \mu(\mu(j,i),\ell) = \ell \}|.$$

To calculate a basis for the radical $R(A)$, we compute the row canonical form $\text{RCF}(\Delta)$ and extract the canonical basis for the nullspace in the usual way: Suppose that Δ has rank r and that the leading 1 of row i of the RCF occurs in column j_i where $1 \leq j_1 < \dots < j_r \leq n$. Let $\Lambda = \{j_1, \dots, j_r\}$ and set $\Phi = X_n \setminus \Lambda$. For each $k = 1, \dots, n-r$ set the $n-r$ free variables x_j ($j \in \Phi$) equal to the k -th unit vector in F^{n-r} and solve for the leading variables x_j ($j \in \Lambda$). We obtain $n-r$ vectors in F^n which form a basis of $R(A)$.

1.6. Structure constants for the semisimple quotient. Let Σ be the $(n-r) \times n$ matrix in which row k contains the coefficients of the k -th radical basis vector. In $\text{RCF}(\Sigma)$, let ℓ_i be the column containing the leading 1 of row i . Row k of $\text{RCF}(\Sigma)$ contains the coefficients of the k -th reduced radical basis vector. Set

$$L = \{\ell_1, \dots, \ell_{n-r}\}, \quad M = \{1, \dots, n\} \setminus L = \{m_1, \dots, m_r\}.$$

The reduced radical basis vectors have the following form for some $\rho_{ij} \in F$:

$$a_{\ell_i} + \sum_{j \in M, j > \ell_i} \rho_{ij} a_j \quad (1 \leq i \leq n-r).$$

We use this reduced basis to compute the structure constants for $Q = A/R$. A basis of Q consists of the cosets $\bar{a}_m = a_m + R$ for $m \in M$. To compute $\bar{a}_i \bar{a}_j$ we observe that $\bar{a}_i \bar{a}_j = \overline{a_i a_j}$, but $a_i a_j$ may contain a_ℓ with $\ell \in L$. These terms must be rewritten using the reduced radical basis relations:

$$\bar{a}_{\ell_i} = - \sum_{m \in M, m > \ell_i} \rho_{im} \bar{a}_m = - \sum_{k=1}^r \sigma_{ik} \bar{a}_{m_k}, \quad \sigma_{ik} = \begin{cases} 0 & \text{if } m_k < \ell_i, \\ \rho_{im_k} & \text{if } m_k > \ell_i. \end{cases}$$

Because we are using the reduced basis, only \bar{a}_m for $m \in M$ occur in $\bar{a}_i \bar{a}_j$. At this point we reindex the basis of Q : we set $b_i = \bar{a}_{m_i}$ for $1 \leq i \leq r$. We have

$$\begin{aligned} b_i b_j &= \bar{a}_{m_i} \bar{a}_{m_j} = \sum_{k=1}^n c_{m_i m_j}^k \bar{a}_k = \sum_{k=1}^r c_{m_i m_j}^{m_k} \bar{a}_{m_k} + \sum_{h=1}^{n-r} c_{m_i m_j}^{\ell_h} \bar{a}_{\ell_h} \\ &= \sum_{k=1}^r c_{m_i m_j}^{m_k} b_k - \sum_{h=1}^{n-r} c_{m_i m_j}^{\ell_h} \sum_{k=1}^r \sigma_{hk} b_k = \sum_{k=1}^r \left(c_{m_i m_j}^{m_k} - \sum_{h=1}^{n-r} c_{m_i m_j}^{\ell_h} \sigma_{hk} \right) b_k. \end{aligned}$$

These structure constants for Q have the following form for some $d_{ij}^k \in F$:

$$b_i b_j = \sum_{k=1}^r d_{ij}^k b_k \quad (i, j = 1, \dots, r).$$

1.7. The center of a semisimple algebra. The next step is to compute the center $Z(Q) = \{x \in Q \mid xy = yx \text{ for all } y \in Q\}$. We quote the following facts:

Theorem 14. [7, Corollary 2.2.8, Theorem 2.4.1] *The center of a semisimple algebra is semisimple. Every commutative semisimple algebra is a direct sum of fields.*

Since Q is the direct sum of simple matrix algebras, and since the center of a simple matrix algebra consists of the scalar matrices, the decomposition

$$Q = Q_1 \oplus \dots \oplus Q_c = M_{n_1}(D_1) \oplus \dots \oplus M_{n_c}(D_c),$$

implies the decomposition $Z(Q) = F_1 \oplus \dots \oplus F_c$ where F_1, \dots, F_c are extension fields of F . Furthermore, $Q_i = QF_i$ for $1 \leq i \leq c$, and this reduces the problem to the commutative case: if we can decompose $Z(Q)$ into the direct sum of fields, then we can decompose Q into the direct sum of simple matrix algebras.

We can represent $Z(Q)$ as the nullspace of a matrix. Let b_1, \dots, b_r be a basis of Q with structure constants d_{ij}^k . Then $x \in Z(Q)$ if and only if $xb_i = b_i x$ for $1 \leq i \leq r$. We have

$$\begin{aligned} x &= \sum_{j=1}^r x_j b_j, & b_i x &= \sum_{j=1}^r x_j b_i b_j = \sum_{j=1}^r x_j \sum_{k=1}^r d_{ij}^k b_k = \sum_{k=1}^r \sum_{j=1}^r d_{ij}^k x_j b_k, \\ x b_i &= \sum_{k=1}^r \sum_{j=1}^r d_{ji}^k x_j b_k, & b_i x - x b_i &= \sum_{k=1}^r \left(\sum_{j=1}^r (d_{ij}^k - d_{ji}^k) x_j \right) b_k. \end{aligned}$$

Corollary 15. *The center $Z(Q)$ is the nullspace of the $r^2 \times r$ matrix in which the entry in row $(i-1)r + k$ and column j is $d_{ij}^k - d_{ji}^k$ for $1 \leq i, j, k \leq r$.*

We compute the RCF of this matrix and the canonical basis of row vectors z_1, \dots, z_c for the nullspace. For $1 \leq i, j \leq c$ we use the structure constants for Q to compute a row vector v_{ij} representing $z_i z_j$ as a linear combination of b_1, \dots, b_r . The coefficients of $z_i z_j$ with respect to the basis z_1, \dots, z_c are the first c entries in the last column of the RCF of the following augmented matrix:

$$\begin{bmatrix} z_1^t & \dots & z_c^t & v_{ij}^t \end{bmatrix}.$$

From this we obtain the structure constants for $Z(Q)$ where $f_{ij}^k \in F$:

$$z_i z_j = \sum_{k=1}^c f_{ij}^k z_k \quad (1 \leq i, j \leq c).$$

1.8. Orthogonal idempotents in a commutative semisimple algebra. Our next task is to decompose the commutative semisimple algebra $Z = Z(Q)$ into a direct sum of fields. We need to find a new basis e_1, \dots, e_c of orthogonal primitive idempotents: $e_i^2 = e_i$ and $e_i e_j = 0$ ($i \neq j$). We use a recursive ideal-splitting procedure following Ivanyos and Rónyai [16]. Let u be a (nonzero) element of a commutative semisimple algebra Z . We compute a basis for the ideal I generated by u , and calculate the identity element of I . We choose a basis element v of I that is not a scalar multiple of the identity element. We compute the minimal polynomial f of v as an element of I , and factor f over F . We have two cases:

- (a) If f is irreducible, then $F(v)$ is a field. If $F(v) = I$ then we are done: the ideal I is a field. If $F(v) \neq I$ then we choose a basis element w of I with $w \notin F(v)$ and compute the minimal polynomial of w over $F(v)$. We repeat this process until we have either (i) constructed a proof that I is a field or (ii) found an element of I whose minimal polynomial is reducible over F .
- (b) If f is reducible, then $f = gh$ where $g, h \in F[x] \setminus F$ are relatively prime. Hence there exist $s, t \in F[x]$ for which $sg + th = 1$. It follows that the ideals J and K generated by $g(v)$ and $h(v)$ split I : that is, J and K are proper ideals of I such that $I = J \oplus K$ and $JK = \{0\}$.

This algorithm starts with $I = Z(Q)$ and recursively performs (a) and (b) to decompose Z into the direct sum of fields. It uses three subprocedures: (1) given a generator of an ideal I , compute a basis of I ; (2) given a basis of I , compute the identity element of I ; (3) given an element of I , compute its minimal polynomial.

For subprocedure (1), we start with an element $u \in Z$. We use the structure constants of Z to compute the products $z_i u$ for $i = 1, \dots, c$. We put these products into the rows of a $c \times c$ matrix and compute its RCF. The nonzero rows of the RCF form a basis of the ideal I generated by u .

For subprocedure (2), let z_1, \dots, z_c be a basis of Z and let I be an ideal with basis y_1, \dots, y_d . We consider an arbitrary $x \in I$ and express y_j in terms of z_k :

$$\begin{aligned}
 xy_k &= \left(\sum_{j=1}^d x_j y_j \right) y_k = \sum_{j=1}^d x_j y_j y_k = \sum_{j=1}^d x_j \sum_{\ell=1}^c y_{j\ell} z_\ell \sum_{m=1}^c y_{km} z_m \\
 &= \sum_{j=1}^d \sum_{\ell=1}^c \sum_{m=1}^c x_j y_{j\ell} y_{km} (z_\ell z_m) = \sum_{j=1}^d \sum_{\ell=1}^c \sum_{m=1}^c x_j y_{j\ell} y_{km} \sum_{p=1}^c f_{\ell m}^p z_p \\
 &= \sum_{p=1}^c \left(\sum_{j=1}^d \left(\sum_{\ell=1}^c \sum_{m=1}^c y_{j\ell} y_{km} f_{\ell m}^p \right) x_j \right) z_p.
 \end{aligned}$$

The conditions $xy_k = y_k$ for $1 \leq k \leq d$ give a linear system of cd equations in the d variables x_1, \dots, x_d :

$$\sum_{j=1}^d \left(\sum_{\ell=1}^c \sum_{m=1}^c y_{j\ell} y_{km} f_{\ell m}^p \right) x_j = y_{kp} \quad (1 \leq k \leq d, 1 \leq p \leq c).$$

The unique solution of this system is the identity element e of the ideal I .

For subprocedure (3), we start with an element $u \in I$, and the previously computed identity element $e \in I$. We represent e as a column vector with respect to the basis z_1, \dots, z_c . Assume that for $j \geq 1$ we have already computed the $c \times j$ matrix whose column vectors are u^{j-1}, \dots, u, e and that this matrix has rank j ; this holds when $j = 1$. We use the structure constants for Z to multiply the first column by u , obtaining u^j ; we then augment the matrix on the left. If this $c \times (j+1)$ matrix has rank $j+1$, we repeat; otherwise, we have a dependence relation among u^j, \dots, u, e , and this is the (not necessarily monic) minimal polynomial. The coefficients of the minimal polynomial are the last column of the RCF.

1.9. Bases for the simple ideals of the semisimple quotient. We now have a new basis e_1, \dots, e_c of orthogonal idempotents in $Z(Q)$; these elements are the identity elements in the extension fields in the decomposition $Z(Q) = F_1 \oplus \dots \oplus F_c$; and these fields are the centers of the simple ideals $Q_i = M_{n_i}(D_i)$ in the

decomposition $Q = Q_1 \oplus \cdots \oplus Q_c$. We have the coefficients of e_1, \dots, e_c with respect to the basis z_1, \dots, z_c of $Z(Q)$, and the coefficients of z_1, \dots, z_c with respect to the basis b_1, \dots, b_r of Q . We obtain elements $e_i \in Q$ (note the ambiguous notation):

$$e_i = \sum_{j=1}^c e_{ij} z_j = \sum_{j=1}^c e_{ij} \sum_{k=1}^r z_{jk} b_k = \sum_{k=1}^r \left(\sum_{j=1}^c e_{ij} z_{jk} \right) b_k.$$

These elements of Q are the identity matrices in the matrix algebras $Q_i = M_{n_i}(D_i)$; they are orthogonal idempotents in Q , but e_i is primitive if and only if $n_i = 1$. We compute a basis of Q_i by constructing a $2r \times r$ matrix; in row j of the upper (resp. lower) $r \times r$ block we put the coefficients of $b_j e_i$ (resp. $e_i b_j$), with respect to b_1, \dots, b_r . We compute the RCF; the nonzero rows form a basis of Q_i .

1.10. Isomorphism of a simple ideal with a full matrix algebra. Suppose that we have a basis s_1, \dots, s_{q^2} and structure constants for an algebra S isomorphic to $M_q(F)$. To construct an explicit isomorphism, we need to find a new basis E_{ij} ($1 \leq i, j \leq q$) satisfying the matrix unit relations $E_{ij} E_{k\ell} = \delta_{jk} E_{i\ell}$. This is easy if we can find a basis for a minimal (q -dimensional) left ideal $I \subset S$: we identify the basis elements of I with the standard basis $U_1, \dots, U_q \in F^q$, and solve the linear equations $E_{ij} U_k = \delta_{jk} U_i$ to determine the elements E_{ij} . If F is finite, then this can be done in polynomial time; but if $F = \mathbb{Q}$, then the problem is more difficult, and seems to be equivalent to hard number-theoretic problems such as integer factorization; see Rónyai [20]. If we are lucky, one of the basis elements of S generates a minimal left ideal; this happens in the example in Part 2.

1.11. Explicit matrices for the irreducible representations. Suppose that we have found an explicit isomorphism of each simple ideal with a full matrix algebra. We then have a new basis of $Q = Q_1 \oplus \cdots \oplus Q_c$ consisting of matrix units:

$$E_{ij}^{(k)} \in Q_k \approx M_{q_k}(F) \quad (1 \leq k \leq c, 1 \leq i, j \leq q_k).$$

Let M be the $r \times r$ matrix which expresses the matrix units $E_{ij}^{(k)}$, ordered in some way, in terms of the original basis: the (ℓ, m) entry of M is the coefficient of b_ℓ in the m -th matrix unit. The inverse matrix expresses the original basis in terms of the matrix units, and has a horizontal block structure: for each $k = 1, \dots, c$ the rows of M^{-1} with indices m from $q_1^2 + \cdots + q_{k-1}^2 + 1$ to $q_1^2 + \cdots + q_k^2$ define the projection of Q onto Q_k . The ℓ -th column of the k -th horizontal block contains the matrix entries in the projection of b_ℓ onto $M_{q_k}(F)$, and from this we obtain the matrix for b_ℓ in the k -th irreducible representation. Composing the map $A \rightarrow A/R = Q$ with the projection $Q \rightarrow Q_k$ gives the matrices representing the basis elements of A .

1.12. Lifting the semisimple quotient to a subalgebra. The last step is to find a subalgebra $B \subseteq A$ which is isomorphic to the semisimple quotient Q and is a vector space complement to the radical R ; the existence of B is guaranteed by the Wedderburn-Malcev Theorem. Let A be an associative algebra of dimension n over F with radical R and semisimple quotient $Q = A/R$. Let $\bar{\beta}_1, \dots, \bar{\beta}_r$ be a basis of Q where $\bar{\beta}_i = \beta_i + R$ with $\beta_i \in A$. We need to find $\gamma_1, \dots, \gamma_r \in R$ so that $\beta_i + \gamma_i \in A$ have the same structure constants $d_{ij}^k \in F$ as $\bar{\beta}_i \in A/R$; that is,

$$(\beta_i + \gamma_i)(\beta_j + \gamma_j) = \sum_{k=1}^r d_{ij}^k (\beta_k + \gamma_k) \quad \text{where} \quad \bar{\beta}_i \bar{\beta}_j = \sum_{k=1}^r d_{ij}^k \bar{\beta}_k.$$

These equations can be rewritten as follows where $\delta_{ij} \in R$:

$$\beta_i \beta_j = \sum_{k=1}^r d_{ij}^k \beta_k + \delta_{ij}, \quad \beta_i \beta_j + \beta_i \gamma_j + \gamma_i \beta_j + \gamma_i \gamma_j = \sum_{k=1}^r d_{ij}^k \beta_k + \sum_{k=1}^r d_{ij}^k \gamma_k.$$

We combine these equations, and consider the special case in which $R^2 = \{0\}$:

$$\beta_i \gamma_j + \gamma_i \beta_j + \gamma_i \gamma_j - \sum_{k=1}^r d_{ij}^k \gamma_k = -\delta_{ij}, \quad \beta_i \gamma_j + \gamma_i \beta_j - \sum_{k=1}^r d_{ij}^k \gamma_k = -\delta_{ij}.$$

The last equation is a linear system in the coefficients $x_{i\ell} \in F$ of the radical terms γ_i with respect to a basis $\zeta_1, \dots, \zeta_{n-r}$ of R . We have

$$\gamma_i = \sum_{\ell=1}^{n-r} x_{i\ell} \zeta_\ell, \quad \sum_{\ell=1}^{n-r} x_{j\ell} \beta_i \zeta_\ell + \sum_{\ell=1}^{n-r} x_{i\ell} \zeta_\ell \beta_j - \sum_{k=1}^r d_{ij}^k \sum_{\ell=1}^{n-r} x_{k\ell} \zeta_\ell = -\delta_{ij}.$$

We expand $\beta_i \zeta_\ell$, $\zeta_\ell \beta_j$ and δ_{ij} in terms of $\zeta_1, \dots, \zeta_{n-r}$ where $\lambda_{i\ell}^t, \rho_{\ell j}^t, \sigma_{ij}^t \in F$:

$$\beta_i \zeta_\ell = \sum_{t=1}^{n-r} \lambda_{i\ell}^t \zeta_t, \quad \zeta_\ell \beta_j = \sum_{t=1}^{n-r} \rho_{\ell j}^t \zeta_t, \quad \delta_{ij} = \sum_{t=1}^r \sigma_{ij}^t \zeta_t.$$

We obtain

$$\sum_{\ell=1}^{n-r} x_{j\ell} \sum_{t=1}^{n-r} \lambda_{i\ell}^t \zeta_t + \sum_{\ell=1}^{n-r} x_{i\ell} \sum_{t=1}^{n-r} \rho_{\ell j}^t \zeta_t - \sum_{t=1}^{n-r} \sum_{k=1}^r d_{ij}^k x_{kt} \zeta_t = -\sum_{t=1}^{n-r} \sigma_{ij}^t \zeta_t.$$

Extracting the coefficient of ζ_t gives

$$\sum_{\ell=1}^{n-r} \lambda_{i\ell}^t x_{j\ell} + \sum_{\ell=1}^{n-r} \rho_{\ell j}^t x_{i\ell} - \sum_{k=1}^r d_{ij}^k x_{kt} = -\sigma_{ij}^t \quad (1 \leq i, j \leq r, 1 \leq t \leq n-r).$$

The terms γ_i are the solution of these $r^2(n-r)$ linear equations in the $r(n-r)$ variables $x_{i\ell}$. This solution is not unique: since any two liftings of the quotient are unipotently conjugate by an element of the form $1 + \zeta$ where $\zeta \in R$ (Theorem 9), the number of parameters will equal the dimension of the radical.

In the general case where $R^2 \neq \{0\}$, suppose that $R^\nu \neq \{0\}$ but $R^{\nu+1} = \{0\}$ for some $\nu \geq 1$; the special case $R^2 = \{0\}$ corresponds to $\nu = 1$. We sketch the approach developed by de Graaf et al. [4] which uses induction on $\mu = 1, \dots, \nu$. The inductive step applies the computations in the special case to compute a lifting of A/R^μ to $A/R^{\mu+1}$ by solving a linear system in the coefficients of terms $\gamma_i \in R^\mu/R^{\mu+1}$ using a basis for a complement of $R^{\mu+1}$ in R^μ . At the last step, when $\mu = \nu$, we have obtained a lifting of A/R to a subalgebra of $A/R^{\nu+1} = A/\{0\} = A$.

2. SEMIGROUPS OF BOOLEAN MATRICES

2.1. Binary relations on a finite set. Perhaps the most general associative structure is the collection of binary relations on a set under the operation of relational composition.

Definition 16. Let n be a positive integer and set $X_n = \{1, \dots, n\}$. The power set $P(X_n^2)$ of the Cartesian square is the collection of all binary relations on X_n . The natural associative operation on $P(X_n^2)$ is relational composition:

$$R \circ S = \{ (i, k) \mid \text{there exists } j \in X_n \text{ such that } (i, j) \in R \text{ and } (j, k) \in S \}.$$

$$\begin{array}{c}
S_n \left| \begin{array}{c} SI_n \\ \sum_{i=1}^n \binom{n}{i}^2 i! \end{array} \right| \begin{array}{c} FT_n \\ n^n \end{array} \left| \begin{array}{c} PT_n \\ (n+1)^n \end{array} \right| \begin{array}{c} HM_n \\ \text{open [12]} \end{array} \left| \begin{array}{c} QP_n \\ \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (2^{n-k}-1)^n \end{array} \right| \begin{array}{c} B_n \\ 2^{n^2} \end{array}
\end{array}$$

TABLE 1. Orders of subsemigroups of the semigroup of binary relations

We represent the relation $R \in P(X_n^2)$ as the $n \times n$ zero-one matrix (m_{ij}) where $m_{ij} = 1$ if and only if $(i, j) \in R$. Relational composition corresponds to matrix multiplication using Boolean arithmetic ($1 + 1 = 1$). This structure is called the **semigroup of binary relations on n elements** and is denoted B_n .

The most familiar subsemigroup of B_n is the **symmetric group** S_n , consisting of all matrices in which each row and each column has exactly one 1. The **symmetric inverse semigroup** SI_n , consisting of all matrices in which each row and each column has at most one 1, corresponds to partial bijections between subsets of X_n . The **full transformation semigroup** FT_n , consisting of all matrices in which each column has exactly one 1, corresponds to functions $X_n \rightarrow X_n$. The **partial transformation semigroup** PT_n , consisting of all matrices in which each column has at most one 1, corresponds to functions from subsets of X_n to X_n . (These four classes are the **classical finite transformation semigroups**; see Ganyushkin and Mazorchuk [14].) The **semigroup of Hall matrices** HM_n consists of all matrices (m_{ij}) which contain a permutation matrix in the sense that for some $\sigma \in S_n$ we have $m_{i, \sigma(i)} = 1$ for $i = 1, \dots, n$. The **semigroup of quasipermutations** QP_n consists of all matrices in which each column and each row has at least one 1. (Every Hall matrix is a quasipermutation, but the converse is false for $n \geq 3$.) These semigroups can be regarded as generalizations of the symmetric group; their orders are given in Table 1.

For the semigroup algebra of a finite semigroup, we have two different bases: first, the elements of the semigroup; second, the matrix units in the Wedderburn decomposition together with the reduced basis of the radical. The projections onto the simple ideals in the semisimple quotient provide irreducible representations of the semigroup; see Bremner and El Bachraoui [1] for a general result regarding B_n .

2.2. The partial transformation semigroup on two elements. We explicitly compute the structure of the semigroup algebra $A = \mathbb{Q}PT_2$ of the semigroup $\{a_1, \dots, a_9\}$ of all 2×2 zero-one matrices in which each column has at most one 1:

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

The multiplication in PT_2 is displayed in Table 2: $a_i a_j = a_{\mu(i,j)}$ where $\mu(i, j)$ is the entry in row i and column j . We study A since (i) it is a small algebra with a nonzero radical in characteristic 0; (ii) there is a unique irreducible representation of dimension > 1 ; (iii) the minimal polynomials of the central elements have rational roots; (iv) the radical has square zero, so we can lift the quotient in one step.

From the multiplication table we obtain the matrix Δ which has the radical R as its nullspace (Corollary 13), and we compute its RCF; see Table 3. The matrix has rank 7, and so R has dimension 2. We set the free variables (x_6, x_9) equal to $(1, 0)$ and $(0, 1)$ to obtain the canonical basis of the nullspace and then the reduced

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 & 6 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 & 3 & 1 & 3 & 2 & 6 \\ 1 & 4 & 5 & 1 & 1 & 9 & 4 & 5 & 1 \\ 1 & 1 & 1 & 4 & 5 & 1 & 5 & 4 & 9 \\ 1 & 2 & 3 & 2 & 3 & 6 & 6 & 6 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 5 & 2 & 3 & 9 & 8 & 7 & 6 \\ 1 & 4 & 5 & 4 & 5 & 9 & 9 & 9 & 9 \end{bmatrix}$$
TABLE 2. Multiplication table for PT_2

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 & 4 & 4 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 1 & 1 & 4 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1 & 4 & 1 & 4 & 1 & 4 & 4 & 4 & 4 \\ 1 & 4 & 1 & 1 & 4 & 4 & 9 & 1 & 4 \\ 1 & 1 & 4 & 4 & 1 & 4 & 1 & 9 & 4 \\ 1 & 1 & 4 & 1 & 4 & 4 & 4 & 4 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & . & . & . & . & -1 & . & . & -1 \\ . & 1 & . & . & . & 1 & . & . & . \\ . & . & 1 & . & . & 1 & . & . & . \\ . & . & . & 1 & . & . & . & . & 1 \\ . & . & . & . & 1 & . & . & . & 1 \\ . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{bmatrix}$$
TABLE 3. The radical matrix for $A = \mathbb{Q}PT_2$ and its row canonical form

$$\begin{bmatrix} 1 & -1 & -1 & . & . & 1 & . & . & . \\ 1 & . & . & -1 & -1 & . & . & . & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & . & . & -1 & -1 & . & . & . & 1 \\ . & 1 & 1 & -1 & -1 & -1 & . & . & 1 \end{bmatrix}$$

TABLE 4. The canonical and reduced bases of the radical of $\mathbb{Q}PT_2$

basis of the radical; see Table 4. The reduced basis consists of these elements of A :

$$\begin{aligned} \zeta_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \\ \zeta_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

We have these corresponding relations in $Q = A/R$:

$$\bar{a}_1 = \bar{a}_4 + \bar{a}_5 - \bar{a}_9, \quad \bar{a}_2 = -\bar{a}_3 + \bar{a}_4 + \bar{a}_5 + \bar{a}_6 - \bar{a}_9.$$

The semisimple quotient Q has dimension 7. We compute the RCF of the matrix whose nullspace is the center, and extract the canonical basis of $Z(Q)$; see Table 5. The center has dimension 4; its structure constants are in Table 6. We need to find a new basis of orthogonal idempotents.

To start, $I = Z(Q)$ with identity element z_2 . Since $z_1^2 = 1$, the minimal polynomial of z_1 is $f = t^2 - t$ and so we take $g = t - 1$ and $h = t$ which gives $I = J \oplus K$ where $J = \langle z_1 - z_2 \rangle$ and $K = \langle z_1 \rangle$. A basis for J (resp. K) is $z_1 - z_2$ and $z_3 - z_4$ (resp. z_1 and z_4). In J the identity element is $-z_1 + z_2$, and $z_3 - z_4$ has minimal polynomial $t^2 - 1$. Hence J splits into 1-dimensional ideals with bases $z_1 - z_2 + z_3 - z_4$

$$\begin{bmatrix} 1 & . & . & 1 & . & 1 & . \\ . & 1 & . & . & . & 1 & 1 \\ . & . & 1 & -1 & . & . & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & . & 1 & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ -1 & -1 & . & . & . & 1 & . \\ . & -1 & -1 & . & . & . & 1 \end{bmatrix}.$$

TABLE 5. RCF of center matrix, and canonical center basis

\cdot	z_1	z_2	z_3	z_4
z_1	z_1	z_1	z_4	z_4
z_2	z_1	z_2	z_3	z_4
z_3	z_4	z_3	$-z_1+z_2-z_4$	$-z_4$
z_4	z_4	z_4	$-z_4$	$-z_4$

TABLE 6. Structure constants for $Z(Q)$

and $z_1 - z_2 - z_3 + z_4$. In K the identity element is z_1 , and z_4 has minimal polynomial $t^2 + t$. Hence K splits into 1-dimensional ideals with bases z_4 and $z_1 + z_4$. Scaling these basis elements so that they satisfy the idempotent equation $e^2 = e$, we obtain

$$e_1 = \frac{1}{2}(-z_1 + z_2 - z_3 + z_4), \quad e_2 = \frac{1}{2}(-z_1 + z_2 + z_3 - z_4), \quad e_3 = -z_4, \quad e_4 = z_1 + z_4.$$

These primitive idempotents in $Z(Q)$ correspond to these elements of Q :

$$\begin{aligned} e_1 &= b_1 - b_3 - \frac{1}{2}b_4 + \frac{1}{2}b_5 - \frac{1}{2}b_6 + \frac{1}{2}b_7, & e_2 &= -\frac{1}{2}b_4 + \frac{1}{2}b_5 + \frac{1}{2}b_6 - \frac{1}{2}b_7, \\ e_3 &= b_2 + b_3 - b_7, & e_4 &= -b_1 - b_2 + b_4 + b_7. \end{aligned}$$

The ideals in Q generated by e_1, e_2, e_3, e_4 have dimensions 1, 1, 1, 4 and so

$$Q = A/R \approx \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}).$$

The 4-dimensional ideal generated by e_4 has basis

$$\alpha = b_1 - b_7, \quad \beta = b_2 - b_7, \quad \gamma = b_3 - b_7, \quad \delta = b_4 - b_7.$$

We need to compute an explicit isomorphism of this ideal with $M_2(\mathbb{Q})$; that is, a new basis E_{ij} which satisfies the matrix unit relations $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$. The dimensions of the left ideals generated by $\alpha, \beta, \gamma, \delta$ are 4, 2, 2, 2. In particular, β generates a 2-dimensional left ideal with basis $U_1 = b_1 - b_3$ and $U_2 = b_2 - b_7$. We identify U_1, U_2 with $(1, 0), (0, 1) \in \mathbb{Q}^2$, and solve for the matrix units; we obtain

$$E_{11} = -b_1 + b_3 + b_4 - b_7, \quad E_{12} = -b_4 + b_7, \quad E_{21} = b_3 - b_7, \quad E_{22} = -b_2 - b_3 + 2b_7.$$

We now have two bases for Q : the old basis b_1, \dots, b_7 and the new basis $e_1, e_2, e_3, E_{11}, E_{12}, E_{21}, E_{22}$. Let M be the matrix whose (i, j) entry is the coefficient of old basis element i in new basis element j . The columns of M express the new basis with respect to the old basis, and hence the columns of M^{-1} express the old basis with respect to the new basis; see Table 7.

Semigroup elements a_1, a_2 are congruent modulo R to linear combinations of a_3, \dots, a_9 ; combining this with M^{-1} we express a_1, a_2 in terms of the matrix units. In this way we express all nine elements of A in terms of the matrix units, and this gives the four irreducible representations; see Table 8. The first two are the unit and sign representations of the symmetric group; the third is the unit representation of the semigroup; the fourth is the irreducible 2-dimensional representation.

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & -1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -1 & -1 & 1 & -1 & 2 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

TABLE 7. Change of basis matrices for Q

element	\mathbb{Q}	\mathbb{Q}	\mathbb{Q}	$M_2(\mathbb{Q})$
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	0	0	1	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0	0	1	$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	0	0	1	$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	0	0	1	$\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	0	0	1	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	0	0	1	$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	1	-1	1	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	0	0	1	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

TABLE 8. Irreducible representations of PT_2

The last step is to find a subalgebra of A isomorphic to the semisimple quotient $Q = A/R$. We use the following ordered basis of A :

$$\beta_1 = e_1, \beta_2 = e_2, \beta_3 = e_3, \beta_4 = E_{11}, \beta_5 = E_{12}, \beta_6 = E_{21}, \beta_7 = E_{22}, \zeta_1, \zeta_2.$$

We compute the quantities δ_{ij} and solve a linear system for the coefficients $x_{i\ell}$ of the terms γ_i for which $\beta_i + \gamma_i$ satisfy the structure constants for Q . We obtain the following matrix, in which row i gives the coefficients of γ_i with respect to the semigroup elements a_1, \dots, a_9 ; the free variables are $\alpha = x_{42}$, $\beta = x_{52}$. The number

$$\begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

TABLE 9. Basis for Wedderburn decomposition of $\mathbb{Q}PT_2$

of parameters is the dimension of the radical, as expected:

$$\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\beta & \frac{1}{2}-\alpha & \frac{1}{2}-\alpha & -\frac{1}{2}+\alpha+\beta & -\frac{1}{2}+\alpha+\beta & -\frac{1}{2}+\alpha & 0 & 0 & \frac{1}{2}-\alpha-\beta \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & \alpha & \alpha & -\alpha & -\alpha & -\alpha & 0 & 0 & \alpha \\ 0 & \beta & \beta & -\beta & -\beta & -\beta & 0 & 0 & \beta \\ \alpha & 0 & 0 & -\alpha & -\alpha & 0 & 0 & 0 & \alpha \\ -1+\beta & 0 & 0 & 1-\beta & 1-\beta & 0 & 0 & 0 & -1+\beta \end{bmatrix}$$

We add the terms γ_i to the original coset representatives β_i to obtain a lifted basis of a subalgebra of A isomorphic to Q . We include the radical basis elements ζ_i to obtain a new basis of A . Choosing $\alpha = 1$, $\beta = 0$ gives the basis for A in Table 9. We now have the complete decomposition of the semigroup algebra $A = \mathbb{Q}PT_2$.

2.3. Further computations. The Maple program used to decompose $\mathbb{Q}PT_2$ can also be used to decompose $\mathbb{Q}PT_3$ and $\mathbb{Q}PT_4$. We obtain the following results:

A	$\dim A$	$\dim R$	$\dim Q$	structure of Q (matrix sizes)
$\mathbb{Q}PT_2$	9	2	7	1, 1, 1, 2
$\mathbb{Q}PT_3$	64	30	34	1, 1, 1, 2, 3, 3, 3
$\mathbb{Q}PT_4$	625	416	209	1, 1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 8

2.4. A constructive approach to the structure of algebras. Since the original paper of Friedl and Rónyai [13], there has been much research on polynomial-time algorithms for explicit computation of the structure of finite-dimensional associative algebras and Lie algebras. In addition to the references already cited, the work of Eberly and Giesbrecht [8, 9, 10, 11] deserves particular mention.

2.5. Representation theory of finite semigroups. There is a substantial literature on the structure theory of semigroup algebras of finite semigroups; the classical reference is Clifford and Preston [2]. A recent monograph is Ganyushkin and Mazorchuk [14]; see also the paper [15]. For the symmetric inverse semigroup, see Solomon [21]. For the full transformation semigroup, see Putcha [18].

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REFERENCES

- [1] M. R. BREMNER, M. EL BACHRAOUI: On the semigroup algebra of binary relations. *Commun. Algebra* (to appear).
- [2] A. H. CLIFFORD, G. B. PRESTON: *The Algebraic Theory of Semigroups*. American Mathematical Society, 1961.
- [3] A. M. COHEN, G. IVANYOS, G. B. WALES: Finding the radical of an algebra of linear transformations. *J. Pure Appl. Algebra* 117/118 (1997) 177–193.
- [4] W. A. DE GRAAF, G. IVANYOS, A. KÜRONYA, L. RÓNYAI: Computing Levi decompositions in Lie algebras. *Appl. Algebra Engrg. Comm. Comput.* 8 (1997) 291–303.
- [5] L. E. DICKSON: *Algebras and Their Arithmetics*. University of Chicago Press, 1923.
- [6] M. P. DRAZIN: Maschke’s theorem for semigroups. *J. Algebra* 72 (1981) 269–278.
- [7] Y. A. DROZD, V. V. KIRICHENKO: *Finite-Dimensional Algebras*. Springer, 1994.
- [8] W. EBERLY: Decomposition of algebras over finite fields and number fields. *Comput. Complexity* 1 (1991) 183–210.
- [9] W. EBERLY: Decompositions of algebras over \mathbb{R} and \mathbb{C} . *Comput. Complexity* 1 (1991) 211–234.
- [10] W. EBERLY, M. GIESBRECHT: Efficient decomposition of associative algebras over finite fields. *J. Symbolic Comput.* 29 (2000) 441–458.
- [11] W. EBERLY, M. GIESBRECHT: Efficient decomposition of separable algebras. *J. Symbolic Comput.* 37 (2004) 35–81.
- [12] C. J. EVERETT, P. R. STEIN: The asymptotic number of $(0, 1)$ -matrices with zero permanent. *Discrete Math.* 6 (1973) 29–34.
- [13] K. FRIEDL, L. RÓNYAI: Polynomial time solutions of some problems in computational algebra. *Proceedings of the 17th Annual ACM Symposium on Theory of Computing*. Association for Computing Machinery, 1985, pages 153–162.
- [14] O. GANYUSHKIN, V. MAZORCHUK: *Classical Finite Transformation Semigroups: An Introduction*. Springer, 2009.
- [15] O. GANYUSHKIN, V. MAZORCHUK, B. STEINBERG: On the irreducible representations of a finite semigroup. *Proc. Amer. Math. Soc.* 137 (2009) 3585–3592.
- [16] G. IVANYOS, L. RÓNYAI: Computations in associative and Lie algebras. Chapter 5 of *Some Tapas of Computer Algebra*. Springer, 1999, pages 91–120.
- [17] K. H. PARSHALL: Joseph H. M. Wedderburn and the structure theory of algebras. *Arch. Hist. Exact Sci.* 32 (1985) 223–349.
- [18] M. PUTCHA: Complex representations of finite monoids. *Proc. London Math. Soc.* (3) 73 (1996) 623–641.
- [19] L. RÓNYAI: Computing the structure of finite algebras. *J. Symbolic Comput.* 9 (1990) 355–373.
- [20] L. RÓNYAI: Simple algebras are difficult. *Proceedings of the 19th Annual ACM Symposium on Theory of Computing*. Association for Computing Machinery, 1987, pages 398–408.
- [21] L. SOLOMON: Representations of the rook monoid. *J. Algebra* 256 (2002) 309–342.

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